Eur. J. Mech. B/Fluids 21 (2002) 91–103

Approximate solutions to the problem of stationary shear flow of smooth granular materials

Alexander V. Bobylev [a], Maria Groppi [b], Giampiero Spiga [b,*]

[a] *Department of Engineering Sciences, Physics and Mathematics, Karlstad University, Universitetgatan 1, 65188 Karlstad, Sweden*
[b] *Dipartimento di Matematica, Università di Parma, Via D'Azeglio 85, 43100 Parma, Italy*

Received 21 February 2001; received in revised form 4 July 2001; accepted 15 August 2001

Abstract

The inelastic Boltzmann equation is used in order to study stationary shear flows in a rarefied granular gas of hard spheres. We resort to a Gaussian moment approximation in order to calculate the pressure tensor in three dimensions. The method is discussed along with previously introduced techniques: asymptotic expansion in the near elastic limit, and pseudo-Maxwellian approximation. Numerical results and approximate analytic formulas for the pressure tensor are presented and briefly commented on. A comparison with the pseudo-Maxwellian solution is discussed in detail. © 2002 Éditions scientifiques et médicales Elsevier SAS. All rights reserved.

Keywords: Granular material; Inelastic Boltzmann equation; Shear flow

1. Introduction

Theoretical and experimental studies of rapid flows of granular materials have become extremely popular in recent years because of their well known industrial applications (see [1], which contains more than 50 references, for a review). The simplest model of such a material is a collection of identical spherical particles interacting by instantaneous collisions. Such a model is very close (at least in the rarefied case) to a classical gas of hard spheres, the main difference being that collisions do not preserve energy. The analogy obviously leads to the idea of using Boltzmann-like kinetic equations, based, however, on modified collision laws. The corresponding inelastic Boltzmann equation is well known and was previously used by many authors for describing rarefied 'granular gases' [1].

We study below a special class of solutions to this nonlinear kinetic equation, the so-called 'stationary shear flow'. In terms of hydrodynamics, such a flow is characterized by constant density, constant temperature (for brevity, we always speak of temperature instead of 'granular temperature'), and constant velocity gradient directed perpendicular to the velocity itself. It is well known that the incompressible Navier–Stokes equations possess solutions which correspond to stationary shear flow. On the other hand, it is easy to verify that the corresponding solution does not exist for classical (elastic) gases described by Boltzmann or (compressible) Navier–Stokes equations, because the energy equation can not be satisfied. The situation, however, changes in the inelastic case, and the solution does exist at both kinetic and hydrodynamic level. This was noticed long ago, and the stationary shear flow for inelastic Boltzmann equation was studied by several authors (see [1,2] and references therein).

A recent publication is due to Cercignani [3], who constructed an approximate solution for the pressure tensor on the basis of the pseudo-Maxwellian model of the inelastic Boltzmann equation introduced in [4] (his solution is exact in the frame of this model). The advantage of Cercignani's solution, compared to previous results obtained by Goldhirsch and his co-authors [1,2], is that no assumption of 'small' inelasticity is made. On the other hand, the pseudo-Maxwellian model itself contains an uncertain

* Correspondence and reprints.
E-mail address: giampiero.spiga@unipr.it (G. Spiga).

PII: S0997-7546(01)01162-1

factor in front of the collision integral, and is based on some non-rigorous 'physical' considerations [4]. Thus, Cercignani's results need a comparison with results obtained by other methods.

Such a comparison is one of the goals of the present paper. The alternative approach is rather obvious: constructing a 'moment approximation' based on the Gaussian distribution function (Gaussian closure [5]). In fact, a similar approach was used earlier by Jenkins and Richman for a two-dimensional gas [6]. We shall see, however, that the realistic three-dimensional case shows an interesting peculiarity of the pseudo-Maxwellian model: the model gives reasonable qualitative results, but introduces a certain degeneracy in the pressure tensor. The main aim of this paper is to consider different approximations to the solution and to compare corresponding results for the pressure tensor.

The paper is organized as follows. In Section 2 we introduce the kinetic equation and make some necessary calculations for second order moments of the collision integral. Then, in Section 3, we formulate the problem and reduce it to a nonlinear eigenvalue problem in dimensionless variables. Such an approach allows us to clarify the mathematical nature of the problem and to compare different techniques from a unified point of view. A perturbation scheme (modified Goldhirsh approach) based on a suitable scaling of the equation is described in Section 4. Equations for second order moments (pressure tensor) are studied in Sections 5 and 6. The exact solution to the pseudo-Maxwellian model equations (Cercignani's solution in slightly different terms) is constructed and discussed in Section 5. The 'Gaussian approximation' (3d analogue of the 2d Jenkins–Richman solution) is studied and solved numerically in Section 6, where we also compare in detail the two alternative approaches. In Section 7 we present the different approximations of the pressure tensor and compare them in the general spirit of rational mechanics [7]. This makes very clear the above mentioned degeneracy of the pseudo-Maxwellian model. On the other hand, the simple analytical formulas for this model are in good agreement, at least qualitatively, with the more general and complex Gaussian approximation. Finally, we construct simple explicit expressions for the Gaussian approximation which reproduce quite accurately our numerical results.

2. Kinetic equation

The simplest model of inelastic collision of two spheres with diameter d can be described as follows. Let $\mathbf{v} \in \mathbb{R}^3$ and $\mathbf{w} \in \mathbb{R}^3$ be velocities of the particles before collision, $\hat{\mathbf{\Omega}} \in \mathbf{S}^2$ be a unit vector pointing from the center of one particle to the center of the other when they collide. Then the velocities $\mathbf{v}'$ and $\mathbf{w}'$ after collision are given by [4]

$$\begin{aligned} \mathbf{v}' &= \frac{1}{2}(\mathbf{v}+\mathbf{w}) + \frac{\mathbf{u}'}{2}, \qquad \mathbf{w}' = \frac{1}{2}(\mathbf{v}+\mathbf{w}) - \frac{\mathbf{u}'}{2}, \\ \mathbf{u}' &= \mathbf{u} - (1+e)(\mathbf{u}\cdot\hat{\mathbf{\Omega}})\hat{\mathbf{\Omega}}, \qquad \mathbf{u} = \mathbf{v}-\mathbf{w}, \qquad \mathbf{u}' = \mathbf{v}'-\mathbf{w}', \end{aligned} \tag{1}$$

where $0 < e \leqslant 1$ is the restitution coefficient. It is sufficient for our goals to consider the case of constant e.

We denote as usual by $f(\mathbf{x},\mathbf{v},t)$ the one-particle distribution function ($\mathbf{x} \in \mathbb{R}^3$ and $t > 0$ stand respectively for position and time variables). Then the Inelastic Boltzmann Equation (IBE) for the above model can be written [1]

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{x}} = Q(f,f) = d^2 \int\limits_{\mathbb{R}^3\times\mathbf{S}^2} \mathrm{d}\mathbf{w}\,\mathrm{d}\hat{\mathbf{\Omega}}(\mathbf{u}\cdot\hat{\mathbf{\Omega}})_+ \left[\frac{1}{e^2} f(\mathbf{v}_*) f(\mathbf{w}_*) - f(\mathbf{v}) f(\mathbf{w})\right], \tag{2}$$

where $\mathbf{v}_*$ and $\mathbf{w}_*$ denote pre-collision velocities associated to the collision mechanism (1). The symbol $(\mathbf{u}\cdot\hat{\mathbf{\Omega}})_+$ coincides with $\mathbf{u}\cdot\hat{\mathbf{\Omega}}$ when such scalar product is non-negative, whereas it takes the value zero whenever the scalar product is negative. Consequently, angular integration ranges over half unit sphere only. The expression for the collision integral $Q(f,f)$ becomes almost obvious if we consider its weak form. Let $g(\mathbf{v})$ be a 'good' test function. Then, after usual transformations, we obtain

$$\begin{aligned} (g,Q) &= \int\limits_{\mathbb{R}^3} \mathrm{d}\mathbf{v}\, g(\mathbf{v})\big[Q(f,f)\big](\mathbf{v}) \\ &= d^2 \int\limits_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbf{S}^2} \mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{w}\,\mathrm{d}\hat{\mathbf{\Omega}}(\mathbf{u}\cdot\hat{\mathbf{\Omega}})_+ f(\mathbf{v}) f(\mathbf{w})\big[g(\mathbf{v}') - g(\mathbf{v})\big], \end{aligned} \tag{3}$$

where $\mathbf{u}$ and $\mathbf{v}'$ are given by (1). The physical meaning of this equality is quite clear: the change of g in a collision integrated over all possible collisions per unit time (at fixed space-time point $(\mathbf{x},t)$, these arguments being omitted for brevity).

Equality (3) can be transformed into a symmetric form

$$(g,Q) = \frac{d^2}{2} \int\limits_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbf{S}^2} \mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{w}\,\mathrm{d}\hat{\mathbf{\Omega}}(\mathbf{u}\cdot\hat{\mathbf{\Omega}})_+ f(\mathbf{v}) f(\mathbf{w})\big[g(\mathbf{v}') + g(\mathbf{w}') - g(\mathbf{v}) - g(\mathbf{w})\big], \tag{4}$$

where notations (1) are used again. The identity [4]

$$\int_{\mathbf{S}^2} d\hat{\boldsymbol{\Omega}}(\mathbf{u}\cdot\hat{\boldsymbol{\Omega}})_+\varphi\big[\hat{\boldsymbol{\Omega}}(\mathbf{u}\cdot\hat{\boldsymbol{\Omega}})\big] = \frac{|\mathbf{u}|}{4}\int_{\mathbf{S}^2} d\hat{\mathbf{n}}\,\varphi\left(\frac{\mathbf{u}-|\mathbf{u}|\hat{\mathbf{n}}}{2}\right) \tag{5}$$

for any function φ leads to

$$(g, Q) = \frac{d^2}{8}\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbf{S}^2} d\mathbf{v}\,d\mathbf{w}\,d\hat{\mathbf{n}}\,|\mathbf{u}|f(\mathbf{v})f(\mathbf{w})\big[g(\mathbf{v}')+g(\mathbf{w}')-g(\mathbf{v})-g(\mathbf{w})\big], \tag{6}$$

where, in the new notation,

$$\begin{aligned}
\mathbf{v}' &= \mathbf{U}+\frac{1}{2}\mathbf{u}', \qquad \mathbf{w}' = \mathbf{U}-\frac{1}{2}\mathbf{u}', \qquad \mathbf{U} = \frac{1}{2}(\mathbf{v}+\mathbf{w}),\\
\mathbf{u} &= \mathbf{v}-\mathbf{w}, \qquad \mathbf{u}' = \mathbf{v}'-\mathbf{w}' = \frac{1}{2}(1+e)|\mathbf{u}|\hat{\mathbf{n}}+\frac{1}{2}(1-e)\mathbf{u}.
\end{aligned} \tag{7}$$

Remark 2.1. The identity (6) gives, as a by-product, a simple rule for Monte-Carlo simulation schemes: take uniformly distributed (on $\mathbf{S}^2$) vector $\hat{\mathbf{n}}$ and change the relative velocity from $\mathbf{u}$ to $\mathbf{u}'$ (see (7)). This rule allows us to use existing computer codes for 'elastic' Boltzmann equation (after minor modifications) in order to simulate rarefied gases with inelastic collisions.

We shall need in the following the second-order moments of the collision integral. The equality (6) with $g = v_i v_j$, $i, j = 1, 2, 3$, leads to

$$\pi d^2 q_{ij} = (v_i v_j, Q) = \frac{d^2}{16}\int_{\mathbb{R}^3\times\mathbb{R}^3} d\mathbf{v}\,d\mathbf{w}\,|\mathbf{u}|f(\mathbf{v})f(\mathbf{w})\int_{\mathbf{S}^2} d\hat{\mathbf{n}}\,(u_i'u_j' - u_i u_j), \tag{8}$$

in notation (7). Then evaluation of the integral over $\mathbf{S}^2$ yields

$$q_{ij} = -\frac{1-\beta^2}{4}\int_{\mathbb{R}^3\times\mathbb{R}^3} d\mathbf{v}\,d\mathbf{w}\,f(\mathbf{v})f(\mathbf{w})|\mathbf{u}|\left[u_i u_j - \theta\frac{|\mathbf{u}|^2}{3}\delta_{ij}\right], \tag{9}$$

where for convenience we have introduced new parameters

$$\beta = \frac{1-e}{2}, \qquad \theta = \frac{1-\beta}{1+\beta} = \frac{1+e}{3-e}, \tag{10}$$

with $1/2 > \beta \geqslant 0$ and $1/3 < \theta \leqslant 1$ for $0 < e \leqslant 1$. Finally, by changing variables from $(\mathbf{v}, \mathbf{w})$ to $(\mathbf{U}, \mathbf{u})$ we obtain, in notation (7),

$$q_{ij} = \frac{1}{\pi d^2}(v_i v_j, Q) = -\frac{1-\beta^2}{4}\int_{\mathbb{R}^3} d\mathbf{u}\,G(\mathbf{u})|\mathbf{u}|\left(u_i u_j - \frac{\theta}{3}|\mathbf{u}|^2\delta_{ij}\right), \tag{11}$$

where

$$G(\mathbf{u}) = \int_{\mathbb{R}^3} d\mathbf{U}\,f\left(\mathbf{U}+\frac{\mathbf{u}}{2}\right)f\left(\mathbf{U}-\frac{\mathbf{u}}{2}\right). \tag{12}$$

This equality will be used in Section 5 in order to construct an approximate solution to the problem stated in the next section.

3. Stationary shear flow

We consider a special class of stationary solutions to the IBE (2) defined by

$$f(\mathbf{x}, \mathbf{v}, t) = F(\mathbf{v}-\boldsymbol{\Pi}\cdot\mathbf{x}), \qquad (F, \mathbf{v}) = \int_{\mathbb{R}^3} d\mathbf{v}\,\mathbf{v}F(\mathbf{v}) = \mathbf{0}, \tag{13}$$

where $\mathbf{\Pi}$ is a constant 3×3 matrix. It is easy to check that such a solution can exist only for a nilpotent matrix $\mathbf{\Pi}$ such that $\mathbf{\Pi}^2 = \mathbf{0}$. Then, by using only rotations and reflections, one can find a coordinate system where [3]

$$\mathbf{\Pi} = k\mathbf{N}, \quad \mathbf{N} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad k = \text{const} > 0. \tag{14}$$

Eq. (2) is invariant under rotations and reflections. Therefore we assume, without loss of generality, that the matrix $\mathbf{\Pi}$ in (13) is given by (14). In Cartesian coordinates

$$\mathbf{x} = (x_1, x_2, x_3), \qquad \mathbf{v} = (v_1, v_2, v_3) \tag{15}$$

the function $F(\mathbf{v})$ in (13) must satisfy the equation

$$-kv_1 \frac{\partial F}{\partial v_2} = Q(F, F), \qquad (F, \mathbf{v}) = \mathbf{0}. \tag{16}$$

The physical meaning of the non-negative solution (provided it exists) is quite clear: it describes a stationary flow with bulk velocity $\mathbf{V}(\mathbf{x}) = (0, kx_1, 0)$ directed along the x_2 axis and having linear profile with respect to the x_1 coordinate (linear shear flow). It is well known [8] that such a stationary solution cannot exist for classical (elastic) Boltzmann equation. We assume that there exists a non-negative solution $F(\mathbf{v})$ of (16) with inelastic collision integral (2), and study below some of its properties.

First of all, we transform (16) to dimensionless form. Denoting

$$\rho = \int_{\mathbb{R}^3} d\mathbf{v}\, F(\mathbf{v}), \qquad T = \frac{1}{3\rho} \int_{\mathbb{R}^3} d\mathbf{v}\, |\mathbf{v}|^2 F(\mathbf{v}),$$
$$F(\mathbf{v}) = \rho T^{-3/2} \widetilde{F}(\tilde{\mathbf{v}}), \qquad \tilde{\mathbf{v}} = \mathbf{v} T^{-1/2}, \tag{17}$$

we substitute (17) into (16) and omit tildas. The result reads

$$-\gamma v_1 \frac{\partial F}{\partial v_2} = \Gamma(F, F) = \frac{1}{\pi} \int_{\mathbb{R}^3 \times \mathbf{S}^2} d\mathbf{v}\, d\hat{\mathbf{\Omega}} (\mathbf{u} \cdot \hat{\mathbf{\Omega}})_+ \left[\frac{1}{e^2} F(\mathbf{v}_*) F(\mathbf{w}_*) - F(\mathbf{v}) F(\mathbf{w}) \right], \tag{18}$$

where

$$\gamma = \frac{k}{\rho \pi d^2 T^{1/2}} \tag{19}$$

and the following normalization conditions hold

$$\int_{\mathbb{R}^3} d\mathbf{v}\, F(\mathbf{v}) \begin{Bmatrix} 1 \\ \mathbf{v} \\ |\mathbf{v}|^2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ \mathbf{0} \\ 3 \end{Bmatrix}. \tag{20}$$

The problem (18) and (20) can be considered as a nonlinear eigenvalue problem for the unknown parameter γ (eigenvalue) and function $F(\mathbf{v})$ satisfying conditions (20). When the solution for F and γ is found, then the temperature is related to density by the equality

$$T = \left(\frac{k}{\rho \pi d^2 \gamma} \right)^2. \tag{21}$$

4. Perturbation method

First we consider the near-elastic case

$$0 \leqslant \beta = \frac{1 - e}{2} \ll 1 \tag{22}$$

studied in detail by Goldhirsch and his co-authors (see [1,2] and references therein). They used the term 'Chapman–Enskog expansion' which seems a bit confusing in the stationary case. In order to clarify the nature of their results we describe briefly below a simple perturbation scheme for solving the nonlinear eigenvalue problem (18) and (20) in case (22).

The key idea is to introduce an artificial parameter ε, such that $\varepsilon = 1$ in all final results, and to rewrite Eq. (18) in the following form (see remark at the end of this section for motivation)

$$\varepsilon\gamma(\varepsilon)v_1\frac{\partial F}{\partial v_2} = \Gamma_0(F,F) + \varepsilon^2\Gamma_1(F,F), \tag{23}$$

where $\Gamma_0(F,F) = [\Gamma(F,F)]_{e=1}$ is the usual (elastic) collision integral and $\Gamma_1(F,F) = \Gamma(F,F) - \Gamma_0(F,F)$. Thus, equations (18) and (23) are identical if $\varepsilon = 1$. Then one can use simple Taylor series

$$\gamma(\varepsilon) = \sum_{n=0}^{\infty}\gamma_n\varepsilon^n, \qquad F = \sum_{n=0}^{\infty}F_n\varepsilon^n, \tag{24}$$

with additional conditions

$$\int_{\mathbb{R}^3} d\mathbf{v}\, F_n(\mathbf{v})\begin{Bmatrix}1\\ \mathbf{v}\\ |\mathbf{v}|^2\end{Bmatrix} = \begin{Bmatrix}1\\ \mathbf{0}\\ 3\end{Bmatrix}\delta_{n0}, \quad n = 0,1,\ldots. \tag{25}$$

We obtain in such a way

$$F_0 = M = (2\pi)^{-3/2}\exp\bigl(-|\mathbf{v}|^2/2\bigr) \tag{26}$$

and denote

$$F_n = M\varphi_n, \qquad \Gamma_0(F_0,F_n) + \Gamma_0(F_n,F_0) = ML\varphi_n, \quad n = 1,\ldots, \tag{27}$$

where L is the usual linearized Boltzmann collision operator. We recall that the equation

$$L\varphi = \psi \tag{28}$$

has a unique solution $\varphi = L^{-1}\psi$ only if

$$\left\langle\varphi, \begin{Bmatrix}1\\ \mathbf{v}\\ |\mathbf{v}|^2\end{Bmatrix}\right\rangle = \left\langle\psi, \begin{Bmatrix}1\\ \mathbf{v}\\ |\mathbf{v}|^2\end{Bmatrix}\right\rangle = \begin{Bmatrix}0\\ \mathbf{0}\\ 0\end{Bmatrix}, \tag{29}$$

where the scalar product $\langle g_1, g_2\rangle$ is defined by

$$\langle g_1, g_2\rangle = \int_{\mathbb{R}^3} d\mathbf{v}\, M(\mathbf{v})g_1(\mathbf{v})g_2(\mathbf{v}). \tag{30}$$

The equation for F_1 becomes

$$F_1 = M\varphi_1, \qquad L\varphi_1 = -M^{-1}\gamma_0 v_1\frac{\partial M}{\partial v_2} = \gamma_0 v_1 v_2, \tag{31}$$

and therefore

$$\varphi_1 = \gamma_0 L^{-1}(v_1v_2), \tag{32}$$

where γ_0 is still unknown. Substituting (24) into (23) we obtain for $n = 1,2,\ldots$

$$\sum_{k=0}^{n}\gamma_{n-k}v_1\frac{\partial F_k}{\partial v_2} + \sum_{k=0}^{n+1}\Gamma_0(F_k, F_{n+1-k}) + \sum_{k=0}^{n-1}\Gamma_1(F_k, F_{n-1-k}) = 0. \tag{33}$$

Then we multiply the equations by $|\mathbf{v}|^2$ and integrate over $\mathbb{R}^3$, to get

$$\begin{aligned}&2\gamma_{n-1}\gamma_0\bigl\langle v_1v_2, L^{-1}(v_1v_2)\bigr\rangle\\ &\quad = -2\sum_{k=2}^{n}\gamma_{n-k}\langle v_1v_2,\varphi_k\rangle + \sum_{k=0}^{n-1}\int_{\mathbb{R}^3} d\mathbf{v}\,|\mathbf{v}|^2\Gamma(F_k, F_{n-1-k}), \quad n = 2,\ldots,\end{aligned} \tag{34}$$

where explicit formulas for $F_{0,1}$ and the definition of Γ_1 have been used. Equations (34) are in fact the solvability conditions for (33). Thus, such a condition for F_{n+1} defines uniquely γ_{n-1}. In particular

$$\gamma_0 = \left[-\frac{\lambda}{2}\int_{\mathbb{R}^3} d\mathbf{v}\,|\mathbf{v}|^2\Gamma(M,M)\right]^{1/2}, \tag{35}$$

where the number λ is defined by

$$\frac{1}{\lambda} = -\langle v_1 v_2, L^{-1}(v_1 v_2)\rangle > 0. \tag{36}$$

The integral in (35) can be easily evaluated by using formula (11), since $Q(f, f) = \pi d^2 \Gamma(f, f)$. This yields

$$\gamma_0 = \left[\frac{8\lambda}{\sqrt{\pi}}\beta(1-\beta)\right]^{1/2} = \left[\frac{2\lambda}{\sqrt{\pi}}\left(1-e^2\right)\right]^{1/2} \tag{37}$$

in complete accordance with the result by Goldhirsch et al. By induction, the equation (33) for $F_{n+1} = M\varphi_{n+1}$ becomes $(n = 1, 2, \ldots)$

$$L\varphi_{n+1} = \gamma_n v_1 v_2 + S_{n+1}(F_0, \ldots, F_n, \gamma_0, \ldots, \gamma_{n-1}), \tag{38}$$

where $F_0, \ldots, F_n$ and $\gamma_0, \ldots, \gamma_{n-1}$ are already known and the orthogonality conditions are satisfied. Then

$$\varphi_{n+1} = \gamma_n L^{-1}(v_1 v_2) + \psi_{n+1}, \quad \psi_{n+1} = L^{-1} S_{n+1}, \tag{39}$$

where ψ_{n+1} is uniquely defined whereas γ_n is still unknown. Substituting (39) into (34) (with $n \to n+1$) we obtain $(n = 1, \ldots)$

$$-\frac{2}{\lambda}\sum_{k=1}^{n+1}\gamma_{n+1-k}\gamma_{k-1} = -2\sum_{k=2}^{n+1}\gamma_{n+1-k}\langle v_1 v_2, \psi_k\rangle + D_n,$$

$$D_n = \sum_{k=0}^{n}\int_{\mathbb{R}^3} \mathrm{d}\mathbf{v}\, |\mathbf{v}|^2 \Gamma(F_k, F_{n-k}). \tag{40}$$

This leads to the recurrence formulas

$$\gamma_n = -\frac{1}{2\gamma_0}\left\{\sum_{k=2}^{n}\gamma_{n+1-k}\gamma_{k-1} + \frac{\lambda}{2}D_n - \lambda\sum_{k=2}^{n+1}\gamma_{n+1-k}\langle v_1 v_2, \psi_k\rangle\right\}, \quad n = 1, 2, \ldots, \tag{41}$$

which define uniquely γ_n through $\{F_0, \ldots, F_n, \gamma_0, \ldots, \gamma_{n-1}\}$. This yields in turn a unique solution for $F_{n+1} = M\varphi_{n+1}$, where φ_{n+1} is given by (39). On the other hand the parameter γ_0 and the functions F_0 and F_1 are already known. This completes the induction and proves the following:

Proposition 4.1. *The nonlinear eigenvalue problem* (23) *has a unique formal solution represented by the Taylor series* (24) *satisfying conditions* (25).

The final result of the expansion with respect to the parameter ε can be easily transformed into series with respect to $\beta^{1/2}$, similarly to what was done by Goldhirsch et al. in the papers quoted above.

Remark 4.2. Our knowledge of previously published results was in fact a motivation for our scaling in Eq. (23). On the other hand, the right scaling can be easily guessed by considering the same equation with $\varepsilon = 1$. Then

$$2\gamma(F, v_1 v_2) = \left(\Gamma_1(F, F), |\mathbf{v}|^2\right), \tag{42}$$

where brackets denote usual scalar product in $L^2(\mathbb{R}^3)$. In the elastic limit we have $F \to M$ and both factors γ and $(F, v_1 v_2)$ tend to zero. In addition, their product must have the same order as $(\Gamma_1, |\mathbf{v}|^2)$. This shows that the right scaling for $\gamma \to 0$ must correspond to the asymptotics $F = M + \mathrm{O}(\gamma)$, $(\Gamma_1(F_0, F_0), |\mathbf{v}|^2) = \mathrm{O}(\gamma^2)$. Therefore we introduce the formal scaling (23) in order to use standard perturbation theory.

We do not discuss practical calculations for the first terms since they are very similar to the ones described in detail by Goldhirsch et al. The only mathematical problem (besides convergence) in this scheme is the construction of a 'good' approximation for L^{-1}, where L is the linearized Boltzmann collision operator for hard spheres. This problem was studied in detail long ago (see, for example, the book [9]), and we do not discuss it here for brevity.

Hence, it is clear how to solve the problem (18) and (20) for the near-elastic case $\beta \ll 1$. The rest of the paper is devoted to the case $\beta = \mathrm{O}(1)$.

5. Moment equations and pseudo-Maxwellian approximation

We consider now the eigenvalue problem (18) and (20) without the assumption (22). It is easy to see that the perturbation solution constructed in section 4 is invariant under reflection $v_3 \to (-v_3)$. Therefore we assume in the following that

$$F(\mathbf{v}) = F\big(v_1, v_2, |v_3|\big). \tag{43}$$

Equations for power moments of $F(\mathbf{v})$ can be obtained by multiplying (18) by the general tensor $v_{i_1}, \dots, v_{i_n}$ and integrating over $\mathbb{R}^3$. Mass and momentum conservations and conditions (20) make cases $n = 0, 1$ trivial. The first nontrivial case is $n = 2$. We denote

$$p_{ij} = \int\limits_{\mathbb{R}^3} \mathrm{d}\mathbf{v}\, v_i v_j F(v), \qquad q_{ij} = \int\limits_{\mathbb{R}^3} \mathrm{d}\mathbf{v}\, v_i v_j \Gamma(F, F) \tag{44}$$

and obtain the following matrix equation

$$\gamma \begin{pmatrix} 0 & p_{11} & 0 \\ p_{11} & 2p_{12} & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} q_{11} & q_{12} & 0 \\ q_{21} & q_{22} & 0 \\ 0 & 0 & q_{33} \end{pmatrix}, \tag{45}$$

where we accounted for the symmetry condition (43). Moreover it is obvious that

$$p_{11} + p_{22} + p_{33} = 3; \quad p_{ij} = p_{ji}, \quad q_{ij} = q_{ji}, \quad i, j = 1, 2, 3, \tag{46}$$

and q_{ij} is given by the integral (11) with $f(\mathbf{v}) = F(\mathbf{v})$. The set (45) of equations for the unknowns p_{ij} is not closed. Any possible (approximate) closure of (45) implies that we express somehow q_{ij} through p_{ij}. After that, we obtain a finite-dimensional nonlinear eigenvalue problem which can be solved analytically or numerically.

An approximate solution for p_{ij} was recently constructed (in slightly different terms) by Cercignani [3] on the basis of pseudo-Maxwellian model of the collision integral (2) introduced in [4]. The general idea of [4] can be used in this specific problem in the following way. We approximate integrals (11) by replacing the factor $|\mathbf{u}|$ (relative speed of colliding particles) by its constant 'average value' $s = \langle |\mathbf{u}| \rangle$. Then we obtain from (18) the approximate equalities

$$q_{ij} \approx -s\frac{1-\beta^2}{4} \int\limits_{\mathbb{R}^3 \times \mathbb{R}^3} \mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{w} F(\mathbf{v}) F(\mathbf{w}) \left(u_i u_j - \frac{\theta}{3} |u|^2 \delta_{ij} \right), \tag{47}$$

with $\mathbf{u} = \mathbf{v} - \mathbf{w}$, and β and θ given by (10).

The simplest way to choose the constant s is to demand the approximate equalities (47) to be exact for $F = M$ (26). This yields

$$s = \frac{16}{3\sqrt{\pi}} \tag{48}$$

and we expect this value to be a reasonable approximation at least for $\beta \ll 1$ (note that $F \to M$ as $\beta \to 0$).

Integrals in (47) can be easily expressed through p_{ij} (44) and we obtain

$$q_{ij} \approx -s\frac{1-\beta^2}{2}(p_{ij} - \theta\delta_{ij}), \tag{49}$$

where conditions (20) are accounted for. This leads to a simple matrix problem

$$\Lambda(\theta) \begin{pmatrix} 0 & p_{11} & 0 \\ p_{11} & 2p_{12} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} p_{11} - \theta & p_{12} & 0 \\ p_{12} & p_{22} - \theta & 0 \\ 0 & 0 & p_{33} - \theta \end{pmatrix} = 0, \tag{50}$$

where $\theta = (1-\beta)/(1+\beta)$ is given, whereas $\Lambda(\theta) = 2\gamma[s(1-\beta^2)]^{-1}$ is unknown. Eq. (50) should be solved under the additional condition (46) $p_{11} + p_{22} + p_{33} = 3$. Moreover $\Lambda(\theta) > 0$, as we assumed from the very beginning (14). One can easily check that Eq. (50) has a unique exact solution

$$\begin{aligned} &p_{11} = p_{33} = \theta, \qquad p_{22} = 3 - 2\theta, \qquad p_{12} = -\Lambda(\theta)\theta, \\ &\Lambda(\theta) = \left[\frac{3(1-\theta)}{2\theta} \right]^{1/2} = \frac{2\gamma}{s(1-\beta^2)}. \end{aligned} \tag{51}$$

It was already mentioned that this solution was first constructed in [3] without specifying the uncertain factor s which is always present in the pseudo-Maxwellian approximation. Besides the simplest approximation (48), the value of s can be chosen in such a way that it yields the correct leading asymptotic term for $\gamma(\beta)$ as $\beta \to 0$ (elastic limit). In such a case

$$s = \left(\frac{32\lambda}{3\sqrt{\pi}}\right)^{1/2}, \qquad \frac{1}{\lambda} = -\langle v_1 v_2, L^{-1}(v_1 v_2)\rangle, \tag{52}$$

as it follows from comparison of formulas (37) and (51) when $\beta \to 0$.

It is remarkable, however, that the diagonal part of the pressure tensor (in our dimensionless variables) does not depend on the uncertain constant s in the pseudo-Maxwellian approximation. Simple formulas $p_{11} = p_{33} = \theta$, $p_{22} = 3 - 2\theta$ reflect a 'degeneracy' typical for this sort of approximation.

6. Gaussian approximation

We return to the exact non-closed matrix equation (45), where

$$q_{ij} = -\frac{1-\beta^2}{4} \int\limits_{\mathbb{R}^3 \times \mathbb{R}^3} \mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{w}\, F(\mathbf{v}) F(\mathbf{w}) |\mathbf{u}| \left(u_i u_j - \frac{\theta}{3} |\mathbf{u}|^2 \delta_{ij}\right), \tag{53}$$

and use a standard way of moment methods in kinetic theory. In order to do this we must choose a 'reasonable' approximation for $F(\mathbf{v})$ which allows us to close the moment equations (45). One can present several arguments (see, for example [5]) on behalf of the Gaussian distribution function

$$F(\mathbf{v}) = \left((2\pi)^3 \det \mathbf{P}\right)^{-1/2} \exp\left(-\frac{1}{2} \mathbf{v} \cdot \mathbf{P}^{-1} \cdot \mathbf{v}\right),$$
$$\mathbf{P} = \int\limits_{\mathbb{R}^3} \mathrm{d}\mathbf{v} F(\mathbf{v}) \mathbf{v} \otimes \mathbf{v}, \qquad \operatorname{Tr} \mathbf{P} = 3 \tag{54}$$

such that all conditions (20) are satisfied. This is a sufficiently simple distribution function to handle analytically, yet it allows us to model the anisotropy of the stress tensor while having the Maxwellian limit in the case of equilibrium. Moreover, if the pressure tensor $\mathbf{P}$ is such that all moment equations up to the second order are satisfied, then the entropy equation is also automatically satisfied. In fact, this approximation is not new in Physics (see for instance [10]), and was already used for two-dimensional stationary shear flow by Jenkins and Richman [6]. We generalize below their approach to the three-dimensional case. One important advantage is that, with this choice of $F(\mathbf{v})$, the entropy equation for (18) is also automatically satisfied. The pressure tensor $\mathbf{P}$ is considered now as a symmetric 3×3 matrix, $\mathbf{P}^{-1}$ denotes its inverse matrix, and moreover

$$\mathbf{v} \cdot \mathbf{P} \cdot \mathbf{v} > 0, \qquad \operatorname{Tr} \mathbf{P} = p_{11} + p_{22} + p_{33} = 3, \qquad p_{3j} = p_{33}\delta_{3j} = p_{j3}. \tag{55}$$

The integral (53) becomes (see (11))

$$q_{ij} = -\frac{1-\beta^2}{4} \int\limits_{\mathbb{R}^3} \mathrm{d}\mathbf{u}\, G(\mathbf{u}) |\mathbf{u}| \left(u_i u_j - \frac{\theta}{3} |\mathbf{u}|^2 \delta_{ij}\right), \tag{56}$$

where

$$G(\mathbf{u}) = \left((4\pi)^3 \det \mathbf{P}\right)^{-1/2} \exp\left(-\frac{1}{4} \mathbf{u} \cdot \mathbf{P}^{-1} \cdot \mathbf{u}\right). \tag{57}$$

The equality (56) in matrix form becomes

$$\mathbf{Q} = -\frac{8(1-\beta^2)}{\sqrt{\pi}} \left(\mathbf{R} - \frac{\theta \operatorname{Tr} \mathbf{R}}{3} \mathbf{I}\right), \tag{58}$$

where $\mathbf{I}$ is the unit matrix and

$$\mathbf{R} = (\det \mathbf{P})^{-1/2} \int\limits_{\mathbb{R}^3} \frac{\mathrm{d}\mathbf{u}}{4\pi} |\mathbf{u}| \exp\left(-\mathbf{u} \cdot \mathbf{P}^{-1} \cdot \mathbf{u}\right) \mathbf{u} \otimes \mathbf{u}. \tag{59}$$

Noting that $\mathbf{P}$ is symmetric and positive definite, we denote $\mathbf{u} = \mathbf{P}^{1/2} \cdot \mathbf{x}$ and obtain

$$\mathbf{R} = \int_{\mathbb{R}^3} \frac{\mathrm{d}\mathbf{u}}{4\pi} \exp\bigl(-|\mathbf{x}|^2\bigr)\bigl|\mathbf{P}^{1/2} \cdot \mathbf{x}\bigr|\bigl(\mathbf{P}^{1/2} \cdot \mathbf{x}\bigr) \otimes \bigl(\mathbf{P}^{1/2} \cdot \mathbf{x}\bigr) = \mathbf{P}^{1/2} \cdot \mathbf{S} \cdot \mathbf{P}^{1/2}, \tag{60}$$

where

$$\mathbf{S} = \int_{\mathbb{R}^3} \frac{\mathrm{d}\mathbf{x}}{4\pi} \exp\bigl(-|\mathbf{x}|^2\bigr)(\mathbf{x} \cdot \mathbf{P} \cdot \mathbf{x})^{1/2} \mathbf{x} \otimes \mathbf{x}. \tag{61}$$

It is obvious that $\mathbf{S}$ is diagonal in the same coordinate system where $\mathbf{P}$ is diagonal. On the other hand, they are both symmetric. Therefore $\mathbf{S} \cdot \mathbf{P} = \mathbf{P} \cdot \mathbf{S}$ and $\mathbf{R} = \mathbf{P}^{1/2} \cdot \mathbf{S} \cdot \mathbf{P}^{1/2} = \mathbf{P} \cdot \mathbf{S} = \mathbf{S} \cdot \mathbf{P}$. The integral (61) can be simplified in the following way:

$$\begin{aligned}\mathbf{S} &= \int_0^\infty \mathrm{d}r\, r^5 \exp\bigl(-r^2\bigr) \int_{\mathbf{S}^2} \frac{\mathrm{d}\hat{\mathbf{\Omega}}}{4\pi} (\hat{\mathbf{\Omega}} \cdot \mathbf{P} \cdot \hat{\mathbf{\Omega}})^{1/2} \hat{\mathbf{\Omega}} \otimes \hat{\mathbf{\Omega}} \\ &= \int_{\mathbf{S}^2} \frac{\mathrm{d}\hat{\mathbf{\Omega}}}{4\pi} (\hat{\mathbf{\Omega}} \cdot \mathbf{P} \cdot \hat{\mathbf{\Omega}})^{1/2} \hat{\mathbf{\Omega}} \otimes \hat{\mathbf{\Omega}}.\end{aligned} \tag{62}$$

Finally we rewrite equality (58) as

$$\mathbf{Q} = -\frac{8(1-\beta^2)}{\sqrt{\pi}} \biggl(\mathbf{P} \cdot \mathbf{S} - \frac{\theta\, \mathrm{Tr}(\mathbf{P} \cdot \mathbf{S})}{3} \mathbf{I}\biggr). \tag{63}$$

Introducing for brevity the new notations

$$\mu(\theta) = \frac{3\sqrt{\pi}\gamma}{8(1-\beta^2)}, \qquad \mathbf{K} = \begin{pmatrix} 0 & p_{11} & 0 \\ p_{11} & 2p_{12} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{64}$$

we transform Eq. (45) in the following matrix form

$$\mu(\theta)\mathbf{K} + 3\mathbf{P} \cdot \mathbf{S} - \bigl(\theta\, \mathrm{Tr}(\mathbf{P} \cdot \mathbf{S})\bigr)\mathbf{I} = \mathbf{0}, \tag{65}$$

where $\mathbf{S}$ is defined by the integral (62), θ by (10), and $\mathrm{Tr}\,\mathbf{P} = 3$.

Formulas (64) and (65) define a new matrix problem which is the 'Gaussian' approximation of the initial problem (45) and (46). The tensor integral (62) can be evaluated by numerical methods. The connection with the pseudo-Maxwellian approximation (section 5) is obvious: (i) definitions of $\Lambda(\theta)$ and $\mu(\theta)$ are identical if the value (48) is used in formula (51) for Λ; (ii) Eqs. (65) and (50) become identical if the (rough) approximation $\mathbf{P} \approx \mathbf{I}$ is used for evaluating the integral (62) (then $\mathbf{S} \approx (1/3)\mathbf{I}$).

In order to solve numerically the problem (64) and (65) we remark that $\mathbf{K}$ is diagonal in the same coordinate system where $\mathbf{P}$ and $\mathbf{S}$ are diagonal, so that $\mathbf{P} \cdot \mathbf{K} = \mathbf{K} \cdot \mathbf{P}$. This fact, combined with $2\gamma p_{12} = \mathrm{Tr}\,\mathbf{Q} < 0$ (following from (45) and from energy dissipation) yields, for $e < 1$,

$$p_{12} = -\biggl(\frac{p_{11}(p_{22} - p_{11})}{2}\biggr)^{1/2} < 0, \qquad p_{22} > p_{11}. \tag{66}$$

The actual unknowns in (65) are thus only p_{11}, p_{22}, and μ. Taking trace we have $\mu \mathrm{Tr}\,\mathbf{K} = -3(1-\theta)\mathrm{Tr}(\mathbf{P} \cdot \mathbf{S})$, and it proves convenient using $z = \theta \mathrm{Tr}(\mathbf{P} \cdot \mathbf{S})$ as unknown parameter, and introducing a new matrix $\mathbf{H} = -\mathbf{K}/p_{12}$. With

$$\mu = -\Lambda^2 \frac{1}{p_{12}} z, \qquad \Lambda^2 = \frac{3(1-\theta)}{2\theta} \tag{67}$$

(compare to (51)), the problem takes the form

$$3\mathbf{P} \cdot \mathbf{S} = z\bigl(\mathbf{I} - \Lambda^2 \mathbf{H}\bigr). \tag{68}$$

The eigenvalues of $\mathbf{P}$ will be labeled by x, y, and $3 - x - y$; analogously the symbols $\alpha_1, \alpha_2, \alpha_3$ and $h_1, h_2, 0$ will denote the corresponding eigenvalues of the matrices $3\mathbf{S}$ and $\mathbf{H}$, where

$$\alpha_i = \frac{3}{4\pi} \int_{\mathbf{S}^2} \sqrt{x\hat{\Omega}_1^2 + y\hat{\Omega}_2^2 + (3-x-y)\hat{\Omega}_3^2}\, \hat{\Omega}_i^2\, \mathrm{d}\hat{\mathbf{\Omega}} = \alpha_i(x, y) \tag{69}$$

can be easily evaluated numerically by the composite midpoint cubature formula for any given x and y. The unknowns x and y follow from a quadratic equation in terms of p_{11} and p_{22}. The option $x < y$, which seems in agreement with (66), corresponds to a suitable rotation of the reference frame and implies the constraint $h_1 > h_2$. Explicitly

$$\begin{aligned} &p_{11} = \frac{2xy}{x+y}, \qquad p_{22} = \frac{x^2+y^2}{x+y}, \qquad p_{33} = 3 - x - y, \\ &p_{12} = -\frac{y-x}{x+y}\sqrt{xy}, \qquad h_1 = \frac{2x}{y-x}, \qquad h_2 = -\frac{2y}{y-x}. \end{aligned} \tag{70}$$

It is easy now to derive from the diagonal form of (68) three equations for the three unknowns x, y, z. After little algebra they may be cast as

$$\begin{aligned} &\alpha_1 x + \alpha_2 y - 2\big(1+\Lambda^2\big)\alpha_3(3-x-y) = 0, \\ &(\alpha_1+\alpha_2)xy - \alpha_3(3-x-y)(x+y) = 0 \end{aligned} \tag{71}$$

with z decoupled as $z = \alpha_3(3-x-y)$. The numerical problem is reduced to determining the zeroes of a vector field in two dimensions, in the physical region $0 < x < y < 3 - x$, as functions of the only input parameter θ. The pressure tensor $\mathbf{P}$ and eigenvalue μ are then provided by (70) and (67). The zeroes have been searched by means of Newton's method for nonlinear systems. In all the quite many cases that have been run systematically, a unique zero has been found, belonging to the admissible region, starting from any admissible initial guess. A unique solution has then been determined for given θ. This numerical solution is discussed below, at the end of Section 7.

7. Constitutive relations

We analyze first the relation between the pressure tensor $\mathbf{P}$ and the velocity gradient in the stationary shear flow of a granular gas according to the previous approaches. In the general case we have [7]

$$\mathbf{V}(\mathbf{x}) = \mathbf{\Pi}\cdot\mathbf{x} = k\,\mathbf{N}\cdot\mathbf{x}, \tag{72}$$

where $\mathbf{N}$ is a nilpotent matrix having the form (14) in a certain coordinate system. Then the most general formula for the (dimensional) pressure tensor can be written [7]

$$\mathbf{P} = p\big[\alpha\mathbf{I} + \tau\big(\mathbf{N}+\mathbf{N}^T\big) + \sigma_1\mathbf{N}^T\cdot\mathbf{N} + \sigma_2\mathbf{N}\cdot\mathbf{N}^T\big], \qquad \operatorname{Tr}\mathbf{P} = 3p, \tag{73}$$

where the notations of [7] are slightly changed. In our case

$$p = \rho T, \qquad 3\alpha + \sigma_1 + \sigma_2 = 3. \tag{74}$$

The coefficients α, $\sigma_{1,2}$, τ do not depend on the coordinate system and it is sufficient to consider (73) in the reference frame where $\mathbf{N}$ is given by (14), so that

$$\alpha = p_{33}, \qquad \sigma_1 = p_{11} - p_{33}, \qquad \sigma_2 = p_{22} - p_{33}, \qquad \tau = p_{12}, \tag{75}$$

where the components p_{ij} were already studied in the previous sections. We present below results for the invariant coefficients α, $\sigma_{1,2}$, τ. A first observation is that all of them are functions of the parameter θ defined by (10), with $1/3 < \theta < 1$. Moreover $\alpha \to 1$, $\sigma_{1,2} \to 0$, $\tau \to 0$ as $\theta \to 1$.

On the other hand, the eigenvalue $\gamma(\theta)$ for the nonlinear problem (45) establishes a relationship between density ρ and temperature T on one side, and velocity gradient k on the other. If the density is chosen arbitrary, then the temperature is defined, for given ρ, k, and θ, by the equality (21). The constitutive equations are then fully determined by the exact knowledge of the coefficients γ, α, $\sigma_{1,2}$, and τ, and formulas (66), (74) show that only three out of them are independent, say γ and $\sigma_{1,2}$, since the others follow as

$$\alpha = \frac{3-\sigma_1-\sigma_2}{3}, \qquad \tau = -\left[\frac{(\alpha+\sigma_1)(\sigma_2-\sigma_1)}{2}\right]^{1/2}. \tag{76}$$

The first equality is exact, whereas the second is approximate (valid for Gaussian closure and pseudo-Maxwellian approximation).

As we discussed above, there exist at least three ways to construct analytical approximations for the necessary coefficients. The first way is to use the perturbation method of Section 4 in a neighborhood of $\theta = 1$ (elastic limit). This way was discussed in detail (in different terms) by Goldhirsch et al. [1,2]. Therefore we note only that

$$\gamma = \mathrm{O}(\sqrt{1-\theta}), \qquad \sigma_{1,2} = \mathrm{O}(1-\theta), \qquad \alpha = 1 + \mathrm{O}(1-\theta), \qquad \tau = \mathrm{O}(\sqrt{1-\theta}), \tag{77}$$

in the elastic limit $\theta \to 1$.

The general case of moderate and strong inelasticity is more interesting for our goals, since accurate results, either analytical or numerical, are lacking in the literature, at least to our knowledge. In such a case we can use the pseudo-Maxwellian approximation (Cercignani's solution) which leads to the formulas (see Section 5)

$$\gamma = \frac{32}{\sqrt{6\pi}}\frac{\sqrt{\theta(1-\theta)}}{(1+\theta)^2}, \qquad \sigma_1 = 0, \qquad \sigma_2 = 3(1-\theta), \qquad \alpha = \theta, \qquad \tau = \sqrt{\frac{3\theta(1-\theta)}{2}}. \tag{78}$$

This result indicates a remarkable degeneracy of the pseudo-Maxwellian approximation, namely $\sigma_1(\theta) = 0$, or $p_{11} = p_{33}$ in the special coordinate system.

The accuracy of this model can be checked by using the Gaussian moment approximation (generalization of the two-dimensional Jenkins–Richman solution) described in Section 6. This approach involves the numerical solution discussed there and yields the values illustrated in Figs. 1, 2 and 3 (solid lines) in comparison to the pseudo-Maxwellian approximation (dashed lines). More precisely, Fig. 1 reports on the diagonal components of the pressure tensor in the special coordinate system, namely p_{11}, p_{22}, and p_{33}, while Fig. 2 illustrates the frame independent coefficients α and $\sigma_{1,2}$, again relevant to normal stresses. Finally, Fig. 3 plots eigenvalue γ and shear stress $p_{12} = \tau$, whose behavior becomes singular (infinite slope) when $\theta \to 1$. Figures show that the simple approximate formulas (78) are in reasonable agreement with the more complex Gaussian moment approximation, but the quantitative discrepancy becomes non-negligible, especially for $\sigma_{1,2}$ and at high inelasticity. In addition, the degeneracy mentioned above is ruled out by the latter approximation, as shown by the trend of σ_1, which vanishes only in the limit $\theta \to 1$. It is also easy to construct approximate analytical formulas for the parameters $\sigma_{1,2}$ plotted in Fig. 2. Their quite smooth trend suggest that they can be satisfactorily approximated by quadratic polynomials

$$\sigma_i(\theta) = A_i(1-\theta) + B_i(1-\theta)^2, \quad i = 1, 2. \tag{79}$$

The coefficients A_i and B_i have been determined numerically by least-squares optimal fitting, and are given by

$$\begin{aligned} &A_1 = -0.1851, \qquad B_1 = 0.1253;\\ &A_2 = 2.264, \qquad B_2 = 0.4902. \end{aligned} \tag{80}$$

The accuracy turns out to be very good, the absolute error between the actual curve and the interpolant parabola never exceeding 1.6×10^{-3} for σ_1 and 8.2×10^{-3} for σ_2, which correspond to relative errors of 2.44% and 0.47% respectively. In any case, the actual numerical solution for σ_i, $i = 1, 2$, and its quadratic analytical approximation overlap almost completely in the figures (not shown here). The function γ is given numerically versus θ by Table 1.

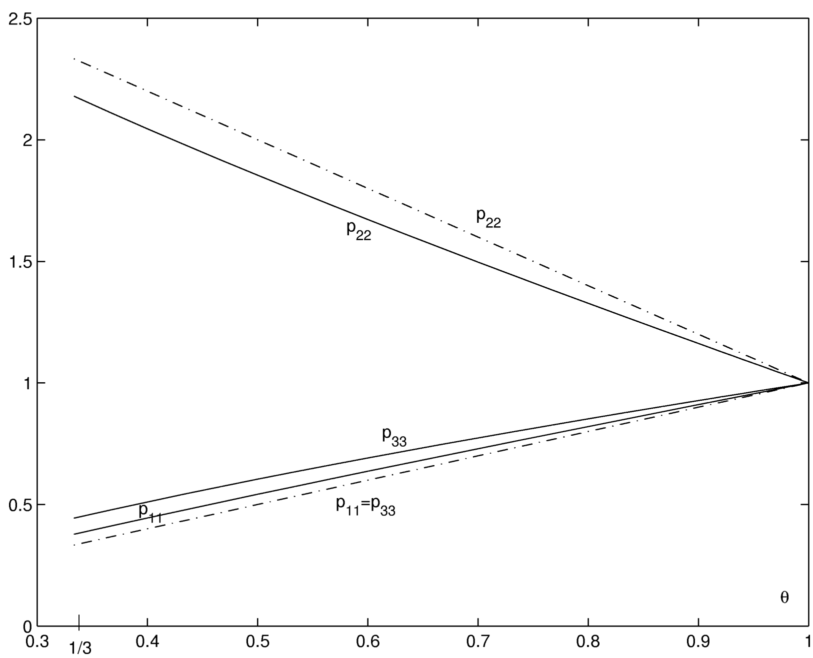

Fig. 1. Diagonal components p_{11}, p_{22} and p_{33} of the pressure tensor versus θ; dashed-dotted curve: pseudo-Maxwellian approximation, solid curve: Gaussian moment approximation.

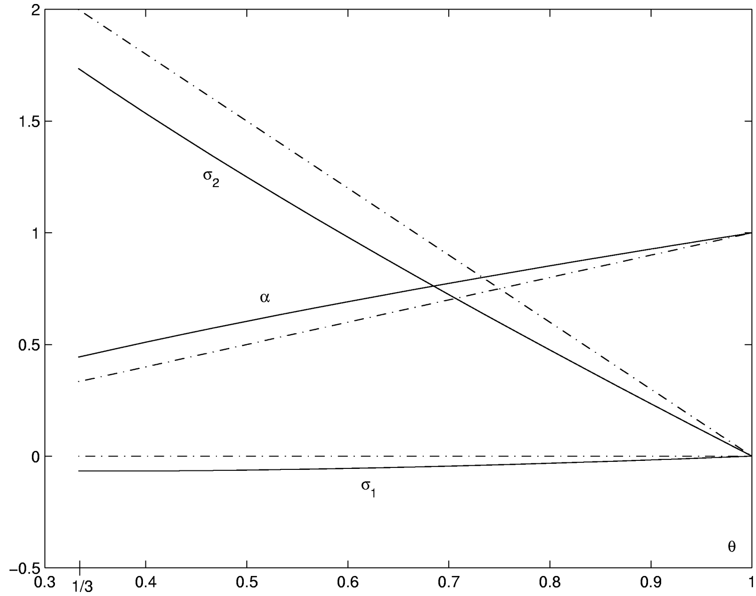

Fig. 2. Invariant quantities α, σ_1 and σ_2 versus θ; dashed-dotted curve: pseudo-Maxwellian approximation, solid curve: Gaussian moment approximation.

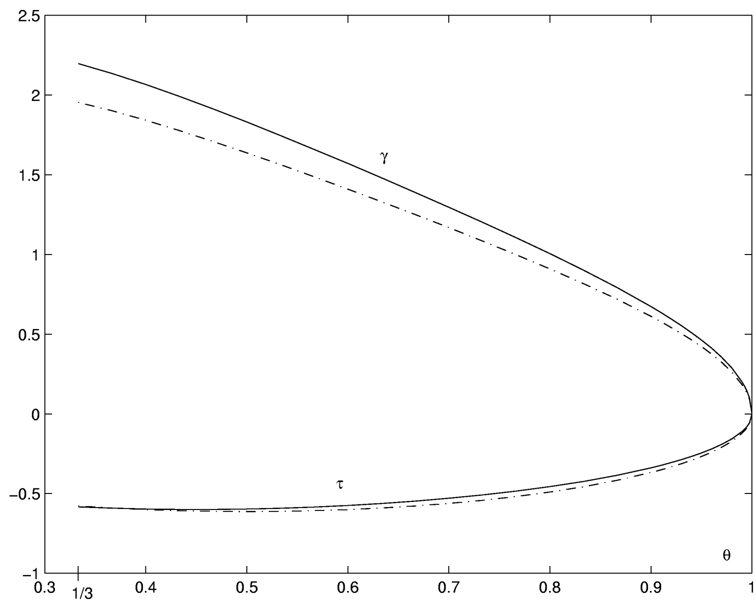

Fig. 3. Shear stress τ and eigenvalue γ versus θ; dashed-dotted curve: pseudo-Maxwellian approximation, solid curve: Gaussian moment approximation.

In conclusion, the pressure tensor is given by:

$$\mathbf{P} = \frac{1}{\rho}\left[\frac{k}{\pi d^2 \gamma(\theta)}\right]^2 \left[\alpha(\theta)\mathbf{I} + \tau(\theta)\left(\mathbf{N} + \mathbf{N}^T\right) + \sigma_1(\theta)\mathbf{N}^T \cdot \mathbf{N} + \sigma_2(\theta)\mathbf{N} \cdot \mathbf{N}^T\right], \tag{81}$$

where (in the Gaussian moment approximation) $\gamma(\theta)$ is given numerically, whereas all other coefficients can be evaluated with good accuracy by formulas (76), (79), and (80).

Table 1
Computed values of eigenvalue $\gamma(\theta)$

θ	1/3	0.40	0.50	0.60	0.70	0.80
$\gamma(\theta)$	2.197	2.067	1.831	1.571	1.297	1.006
θ	0.85	0.90	0.93	0.945	0.960	0.975
$\gamma(\theta)$	0.8749	0.6740	0.5548	0.4878	0.4126	0.3235
θ	0.980	0.985	0.990	0.995	0.998	1.000
$\gamma(\theta)$	0.2886	0.2492	0.2030	0.1431	0.0904	0.000

Acknowledgements

This work has been performed in the framework of the activities sponsored by MURST (Project "Mathematical Problems of Kinetic Theories"), CNR, GNFM, and University of Parma (Italy), by NFR, grant 20005092 (Sweden), and by the European TMR-Network "Asymptotic Methods in Kinetic Theory".

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